Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education

Editors: Uffe Thomas Jankvist, Marja van den Heuvel-Panhuizen, Michiel Veldhuis
Organised by: Freudenthal Group & Freudenthal Institute, Utrecht University
Year: 2019

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Publisher

Freudenthal Group & Freudenthal Institute, Utrecht University, Netherlands and ERME

ISBN 978-90-73346-75-8

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Recommended citation for the proceedings

Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.). (2019). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Recommended citation for single entries in the proceedings

Authors (2019). Title of paper/poster. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. xxxx-yyyy). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Epistemological obstacles in the evolution of the concept of proof in the path of ancient Greek tradition

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The paper examines the epistemological evolution of the concept of proof in the Western tradition, highlighting three important epistemological obstacles whose consideration can have significant consequences on the teaching and learning of proof.

Keywords: Proof, epistemological obstacles, to convince, contradiction, semiotic.

Introduction

Bachelard (1938) identifies in the act of knowing the causes of stagnation of the evolution of scientific thought; he calls the psychological causes of inertia in this evolution *epistemological obstacles*. According to the author (Bachelard, 1938), the epistemologist must endeavor to grasp the scientific concepts in an actual, i.e. in a progressive, psychological syntheses, establishing, concerning every notion, a scale of concepts, which shows how one concept has produced another; moreover, the epistemologist must necessarily take a normative point of view, while the historian usually has to avoid it, and what must attract his attention and guide his research is the search for rationality and construction in the evolution of scientific thought. “The epistemologist must take the facts as ideas, inserting them into a system of thoughts. A fact misinterpreted in an era remains a fact for the historian. It is, at the discretion of the epistemologist, an obstacle, it is a counter-thought” (Bachelard, 1938, p. 17). It is clear that in the construction of the epistemological trajectory it will not be possible to provide (nor would it make sense to ask for) a proof of uniqueness; what such a reconstruction can produce is an argument, effective because of its coherence and its explanatory capacity, which testifies in favour of its one existence.

Brousseau (1989) gives to the concept of epistemological obstacle the meaning of knowledge (not a lack of knowledge) that has been effective previously, in a given context, which at a certain point begins to generate answers judged to be false or inadequate and produces contradictions. Moreover, an epistemological obstacle has the characteristic of being resistant and of presenting itself sporadically even after having been overcome; its overcoming requires the passage to a deeper knowledge, which generalizes the known context and requires that the student becomes explicitly aware of it (Brousseau, 1989). According to Brousseau (1989), this reasoning can be applied to analyse either the historical genesis of knowledge or its teaching or the spontaneous cognitive evolution of student’s conception. The search for epistemological obstacles can occur using two approaches: the first is, according to Bachelard, based on historical research by adopting an epistemological lens; the second one is based on the search for recurring errors in the learning process of a mathematical concept; the two research lines are intertwined: the historical-epistemological evolution can help in the identification of possible hidden models and can provide suggestions on the construction of appropriate learning situations to overcome a given obstacle; on the other hand, students’ errors and recurring difficulties can suggest the presence of epistemological obstacles. Given students' enormous difficulty in understanding proof, it seems to us that the identification of possible epistemological obstacles can be useful in teaching and learning practice and, even before, in teachers' training, providing awareness of teachers' own epistemology and of the possible gap between their own and students' conceptions.

The historical-epistemological investigations on the concept of proof are not new in Mathematics Education. Barbin (1994) traces briefly a history of proof, highlighting its different meanings in different eras. The author identifies two fractures, one relative to the convincing-explaining contrast and one relative to the role of contradiction, but does not clearly highlight the origins of proof as a conviction from an epistemological point of view; moreover, in the identification of the second fracture, she assumes that modern axiomatization is the result of a need for greater proof rigor, while we intend to show that it arises from the very nature of mathematics. Barbin concludes that deduction constitutes an obstacle and that its overcoming can be achieved through the revaluation of proof as a method. Another historical analysis that takes at times interesting epistemological aspects is that of Grabiner (2012), in which the author mainly aims to highlight the non-absoluteness of the concept of rigor and proof, without however proposing an evolutionary trajectory in which, as Bachelard states, it should become clear how one conception follows from the other (Bachelard, 1938). A last contribution that we take briefly into consideration is that of Longo (2012), in which the author emphasizes the pre-eminence of two fundamental principles for mathematical proof, which have cultural and historical roots: symmetry and order, and highlights the need for an approach in opposition to Hilbert's formalism, which should avoid recourse to actual infinity. Longo's approach is interesting but it is posed in a perspective of synthetic philosophy, which is not however the one on which the current foundation of mathematics is based.

None of the contributions examined searches explicitly for epistemological obstacles, taking at the same time a position in line with the current foundational aspects of mathematics, which are those on which teachers' training is based on and which are transposed in textbooks. Our aim is to understand if there is a path in the epistemological evolution of the object proof in the ancient Greek (and then the Western) tradition that could show changes in the modes proof was considered able to provide knowledge and about the characteristics of that knowledge.

The epistemological evolution of the concept of proof

Ancient Greek tradition

One of the most interesting aspects of proof in the path of ancient Greek tradition is related to its origin in an axiomatic system. Even when analyzing various other aspects, Grabiner (2012) identifies at least the reason of what she calls "a logical proof", in Greek mathematicians' need to be able to prove things that are not evident: "[…] visual demonstration did not suffice for the Greeks. […]. Such proofs are necessary when what is being proved is not apparent." (Grabiner, 2012, p. 148). We believe that from an epistemological point of view the answer of the question is a bit more complex and should be searched upstream, in Eleatics' rejection of the epistemological validity of the experience of the senses, which led them to assume logical indicators of clarity and necessity as criteria of truth. We will explain our point of view in the next paragraph, following the original sources.

According to Parmenides, the Being is the only reality and the convincement, in the sense of persuasion through thought, is the only way that allows us to acquire reliable knowledge. Parmenides' conception of thought is that of reasoning, of logos, of what can be called "logical reasoning", which produces *necessary* true conclusions, because when investigating truth (i.e. the Being), we start always from the Being which is not only an unchanging, indestructible whole, but is also connected: “observe how what is far away is reliably approached by your mind because it [the mind] will not separate the Being from the context of Being, nor in such a way that it [the Being] loosen anywhere in its structure nor in such a way that it masses together. Then Being is indeed connected, regardless of where I start [the research] that's where I'll be back again.” (Diels, 1906, p. 116). The last sentence shows exactly the conception of knowledge acquirement in Parmenides: logical reasoning, starting from necessary true premises is convincing and this kind of convincement is the only way that leads to knowledge of truth. Instead the way of investigating the Not-Being, which is not necessary, is the way of opinion linked to the senses: “the only thinkable ways of research are the following: the way of Being, which is and which is impossible but be, is the way of conviction (since it follows the truth); but the other one, that it [the Being] isn’t and that this Not-Being is necessary, is a completely inscrutable path since the Not-Being cannot be neither recognized (it is in fact unfeasible) nor pronounced” (Diels, 1906, p. 116). The fundamental Eleatic dialectic is then the dialectic between certain knowledge (and the way of convincement which allows to acquire it) on the one part, and opinion (and the way of the senses which allows to acquire it) on the other part.

Eleatics used different modes of conviction to carry out Parmenides’ “way of truth”: Zeno of Elea used the proof by *reductio ad absurdum* (or proof by contradiction) as method to destroy the arguments of the opponents showing that their premises led to contradictions, while other Eleatics, like Melissus, tended to prove arguments also starting from undoubtedly certain premises. (Cambiano, 2004). Zeno’s way of *reductio ad absurdum* is coherent with the Eleatic philosophy because there is not only a distinction of Being (real, true) and Not-Being (not real, false), that excludes the possibility of a third value, but also the fact that for the Being not be is impossible, that represents the double negation. Like outlined by Antonini and Mariotti (2008), in the past, starting from 16th and 17th centuries and up to the 20th century, there was a debate about the fact that proof by contradiction collides with the Aristotelian position that causality should be the base of scientific knowledge and that such a proof cannot reveal the causes since it is not based on true premises. We state that in the perspective of Eleatic philosophy, proof by contradiction fits perfectly with the role it has in this philosophy: the role to get knowledge of the truth of a statement by following a convincing logical thought that shows that it is necessarily so and it cannot be in another way. This is just what Zeno probably wanted to do, using such proofs, because they are not suitable to explain, but they are suitable to convince.

The Eleatic dialectic was of great importance for the subsequent development of philosophy and science. Plato’s thinking was strongly influenced by Parmenides (Cambiano, 2004) and so all the Western philosophy.

Other important aspects in the epistemological evolution of proof are the roles played by the Megarian school and by the Sophists. The Megarian philosophical school flourished in the 4th century BCE and some of the successors of Euclid of Megara, its founder, developed logic to such an extent, maintaining the Eleatic epistemological assumptions about the importance of conviction in knowledge acquirement, that they became a school in its own (Cambiano, 2004).

In the 5th century BCE the Greek society experienced a period of rapid socio-cultural transformation that led to a new politic system based on democratic principles; in this new society the need to provide education for the children of the emerging classes arose and Sophists satisfied it (Cambiano, 2004). The ability to convince was considered one of the most important abilities for the citizen who wanted to participate in the political life of the *polis*. Sophists wanted to show that with the art of rhetoric and dialectic, even “weak” speeches could be made “strong”, since it would have been possible to reject even the most evident statements. This was a turning point to relativism with respect to Eleatic doctrine because Sophists negates the existence of absolute truth: truth is a form of knowledge always related to the subject and its experience; they are as multitude of truths as subjects. The dialectic methods used by the Sophists for their argumentations were the same used by the Eleatics: confirmation by proof conduced in rigorous logic steps and refusal by proof of the falsity of the antithesis, but Sophists perfected them; dialectic influenced deeply rhetoric, shifting attention to the persuasive force of speeches.

The Sophists’ relativism pushed many philosophers, e.g. Plato, to look for a transcendent objective basis on which to base behavior and moral conduct but also influenced the way of exposing scientific knowledge, particularly the mathematical one. It becomes clear that the Euclidean axiomatic system was developed at a time when it was necessary to place mathematical knowledge on unquestionable bases. This point of view is clearly expressed for instance by Clairaut: “this geometrician [Euclid] had to convince obstinate Sophists who were proud to reject the most obvious truths” (Clairaut, 1741, pp. 10–11). Hoüel remarks furthermore that Euclid’s frequent recourse to indirect proofs is also motivated by the need to prevent critics due to Sophists’ relativism: “[…] hence his [Euclid's] habit of always proving that a thing cannot be, instead of showing that it is” (Hoüel, 1867, p. 7).

The need to avoid any critic as the tread in organizing the mathematical knowledge can also be detected in the attention placed by Euclid on avoiding arguments which involve concepts that might leave some room for criticism, instead of using direct proofs which involve only subjects conceptually close to the statement to be proved. An example in this sense is the proof of proposition CXVII of book X of the *Elements* which states that in a square the diagonal and the site are incommensurable. In order to prove the statement Euclid uses an indirect proof that involves not only geometrical but also arithmetical arguments (e.g. concepts of odd and even number, of divisibility etc.) instead of using the proposition II of book X, introduced just before, which applies to segments the arithmetical method used in book VII to establish if two numbers are coprime (Barbin, 1994). A proof conducted using the latter method would be direct and would refer only to geometric concepts, but the problem is that it would involve an infinite procedure and might be exposed to the criticism of who asked what is, for example, the limit of the succession of squares that is built.

According to Barbin (1994) we can state that Euclid’s main concern is to be able to persuade and to avoid considerations that might leave room for critiques; an axiomatic system is a good solution to do this. Of course, the methods that could be applied to isolate the axioms might have already been highlighted by Hippocrates of Chios, “who wrote the best logically structured *Elements of Geometry* until Euclid wrote his own *Elements* 150 years afterwards” (Grabiner, 2012, p. 150), but the real motivation to do this should be detected in the philosophical and cultural environment of the time.

In our analysis we cannot avoid to mention Aristotle because of his influence in shaping medieval scholarship, not only in the Western but also in the Islamic world. Aristotle, starting initially from a Platonist position, reversed later in a certain sense the ontological question advocated by the Eleatics and then by his master Plato. Aristotelian ontology, like the Platonic one, distinguishes between

Universal and Particular, but while for Plato universal forms are separate from particular manifestations, for Aristotle the Universal is in the Particular and the only way to access the Universal is through the Particular, using the perception of the senses, in particular the eye, because in perceiving the world we prefer the sight to all the other senses (Aristotle, 1973). One might wonder why the Aristotelian doctrines have not changed the idea of proof and why there is no trace of them in the *Elements* and why the concept of proof contained in the *Elements* remained the prototype of (not only) mathematical proof for a lot of time. We can say that while the ontological aspect is reversed, Aristotle's epistemology remains similar to the Platonic one. Indeed, both, Plato and Aristotle, are interested in the knowledge of Universals though the way to reach them is different: Aristotle's empirical approach is linked to exploration, which can also refer to induction or abduction, but its aim is to arrive at "true" knowledge; this last way is nonetheless deductive and proof is the final step of the process of knowledge acquirement (Cambiano, 2004).

Enlightenment and the "small steps": the first crisis

Until the 17th century the concept of proof in mathematics remained linked to the idea of proof in an axiomatic system such as that of geometry presented in Euclid's *Elements*, even if, due to the lack of adequate mathematical tools, attempts to axiomatize other fields of mathematics remained sporadic and ineffective for a long time (Lolli, 2004). The 17th century was a particular historical and cultural period during which there were a lot of important discoveries and innovative and courageous stances in science. Alongside of Galileo's scientific method, new general mathematical methods were developed in order to satisfy the need to describe the physical phenomena, like Cavalieri's principle of indivisibles, the Cartesian method, Newton's method, Monge's projective method etc. In these rapid innovations the ancient Greek writings, first of all Euclid's *Elements*, were exposed to a hard criticism by the mathematicians. They accused ancient Greek geometer to be more interested in persuasion and in validation than in explanation and discovery. What is criticized is not the content but the form, the setting and the role attributed to proof. In Arnauld's and Nicole's *La logique ou l'art de penser* the authors expose the widespread feeling among their contemporaries summarizing as follow the errors attributed to Greek geometers: (1) to pay more attention to certainty than to evidence and to the conviction of the mind than to its enlightenment; (2) to prove things that have no need of proof; (3) to demonstrate by impossibility; (4) to conduce far-tetched demonstrations; (5) to pay no attention to the true order of nature; (6) to employ no divisions and partitions (Arnauld & Nicole, 1850). The critiques are basically two. The first, which contains the first four errors, concerns the single results and can be summarized in the question: "Is it more important to convince or to enlighten?". The second, which involves the whole structure of the text, is the accusation of a lack of method (the sixth error can be regarded as the consequence of the fifth one).

With regard to the new methods elaborated by mathematicians during the 17th century there was an important epistemological aspect that mathematicians of the time were already asking themselves: the problem of the validity of the new methods. Barbin summarizes as follows the doubt in this regard: "these methods illuminate, clarify the spirit, as they show the way to which we have passed and lead to evidence [...] but what is made evident can be considered as demonstrated?" (Barbin, 1994, p. 223). Thus, for example, Cavalieri's method of indivisibles, which already carried the idea of the integral, aroused some suspicion because of its resort to actual infinity, despite its operational superiority compared to Archimedes' method of exhaustion.

One method that initially does not raise suspicion was instead the Cartesian method which essentially led to a general algebraic conception of proof. The method described by Descartes in his *Discours de la Méthode* (1966), his *small steps*, his *long chains of reasons, all simple and easy,* were the attempt to provide a method of discovery to mathematics which quickly imposed itself on a large scale even though it posed some practical difficulties and mathematicians found it difficult to resort to it in practice (Lolli, 1988). We should however keep in mind that there are two different layers of evidence in Descartes' conception: the one is related to the physical evidence of the axioms in Euclidean geometry, which is the basis of mathematics and that remains undisputed; the other is related to the evidence of the starting point and of the "small steps" of reasoning that should be so simple and evident to be intuitively accepted by anybody without any doubt about their correctness. Both these evidences will be questioned by the subsequent development of mathematics.

Mathematical proofs, physical reasons and the second crisis

At the end of 17th and beginning of 18th centuries, with the growing of Analysis, it becomes more and more clear that the algebraic conception of proof was insufficient for the type of investigation mathematics was going to face [see for instance the need to deal with functions and infinitesimals and in general with infinite entities (Lolli, 1988)] and that requires a purely mathematical definition of the involved objects. Furthermore, the development of abstract algebra allowed a new, structural organization of mathematical knowledge. All these reasons allow the take-off of pure mathematics which does not (should not and could not) refer to physical evidence. On the other hand, the birth of non-Euclidean geometries led to a subsequent loss of authority of geometry as foundation of mathematics, based before on the physical evidence of its axioms. The birth of general theories with multiple interpretations and the need to give a unitary organization of mathematics led to increasing awareness of the importance of the definition of mathematical objects trough interrelations of formal axioms, which allows multiple interpretations; all this finally led to the modern axiomatic model.

Axiomatization implies an important change of perspective: a change in concept and role of contradiction: in ancient Greek mathematics the contradiction intervened in a social act and was used to convince: to prove means to convince; the modern conception of contradiction is very different: contradiction intervenes in a system of mathematical propositions and it is used for producing mathematical results: to prove means to make evident the non-contradictoriness of a statement within an axiomatic system. The concept of evidence makes no sense in this formalistic conception of mathematics: the objects are *real* inasmuch they are defined by the relations between them; there is no need for physical or metaphysical reality to refer to. The only way to deal with such abstract formal objects is formal reasoning: implicit definition and formal logical principles are needed for the new conception of proof, based on non-contradictoriness; the new axiomatic model is not due to a perverse and arbitrary will of formalization but to the formal character of mathematics (Lolli, 2015).

Modern axiomatization and the need for interpretation

Modern axiomatization underpins and highlights the special ontological condition of mathematical objects to which an ostensive cross-reference is impossible and shows clearly the essentially semiotic nature of mathematical activity; but it poses an important problem: in a formal axiomatic conception of mathematics asking for meaning of mathematical objects makes no sense. The only way to give meaning to them is dealing with them in an interpretation of the formal theory. An interpretation needs a language that allows to express the relations of the objects occurring in it. Bourbakism rendered clear that set theory is a formal language which is most similar to natural language and that allows to construct mathematical discourses, giving a meaning to mathematical activity. Experts in mathematics are able to give a meaning to a discourse in set-theoretical language (they are able to use it, enriching its expressivity by natural language without losing its mathematical meaning) and so they are usually convinced of the unity between mathematical and natural languages; but this competence is only the confirmation of the fact that the metalanguage (natural language) has successfully fulfilled its role as constructing tool for the object language (mathematical language).

Conclusions

Our analysis shows three epistemological obstacles which forcefully impose themselves in the examined path: (1) the belief that the proof concept in Euclidean Geometry may be used to *explain* mathematical reasoning; (2) the belief that contradiction in mathematics has the role of convincing and that to prove means to convince somebody of the truth of an utterance and not the role to establish the consistency in an axiomatic system, i.e. to establish its validity; (3) the supposition that mathematics is not a discourse in itself but one that is telling something about something that really exists and that to prove means to prove the truth of a statement and not it's validity in a metalanguage as well the subsequent supposition of the unity between natural language and mathematical language and the consequent lack of awareness of the complex relation between them.

All three obstacles represent important topics related to the teacher's epistemological believes and the way in which teachers implement the didactic transposition of the mathematical object "proof". Moreover, we stress that against the third obstacle, the whole teaching of modern mathematics collides. Without the awareness of the latter, at school we will continue to teach only Euclidean mathematics and the concept of proof present in it, believing that the modern axiomatic method is only a generalization of the Euclidean one.

Furthermore, while the first two obstacles can be considered as obstacles already crystallized in Western culture, the third is an obstacle of very recent formation and it is still little perceived and not recognized. This obstacle also places the cultural aspect in the foreground as it can have very different cultural declensions. Indeed, while the general semiotic aspects might have a general characterization, semiotic aspects related to the natural language can be specific (Lolli, 2015). While the overcoming of the first two obstacles requires the entry into a certain culture of mathematical thought (which has its roots in ancient Greek tradition), the overcoming of the third obstacle requires a specific transposition, dependent on the peculiarities of the natural language of reference.

Concerned to the usefulness of the epistemological obstacles in Mathematics Education, we can say that within Brousseau's (constructivist) theory of didactic situations, acquiring knowledge means adapting to a specially designed situation that has that given knowledge as optimal solution (Brousseau, 1989); it is therefore worth to identify the epistemological obstacles in order to help teachers to construct situations in which the student is forced to use a type of knowledge that leads her/him to overcome the obstacles, supported by teacher's mediation choices. In a socio-cultural perspective, instead, the interpretation of the notion of epistemological obstacle could present some difficulty because of the specificity of mathematical cultural production (D'Amore, Radford, & Bagni, 2006). Nevertheless, also in a socio-cultural perspective, the institutional meaning of mathematical objects refers to a certain cultural tradition which could hide eventual epistemological obstacles.

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